DEPARTMENT OF PHYSICS, NATIONAL CHUNG CHENG UNIVERSITY
QUALIFYING EXAM, FALL 2016
QUANTUM MECHANICS

(1) A beam of spin-1/2 particles is sent through a sequence of three Stern-Gerlach (SG) analyzers (see the figure below), where the first and the third analyzers are aligned along the z-direction, while the second along a direction $\hat{n}$ that makes an angle θ in the xz-plane with respect to the z-axis. Note that the first two SG analyzers are designed to allow the spin-up component for each device orientation to pass.

(a) (10 points) What is the probability that particles transmitted through the first SG analyzer are measured to have spin down at the third SG analyzer?

(b) (10 points) How must the angle θ of the second SG analyzer be oriented so as to maximize the probability that particles are measured to have spin down at the third SG analyzer? What is this maximum fraction?

(c) (5 points) What is the probability that particles have spin down at the third SG analyzer if the second SG analyzer is removed from the experiment?

(*Hint*: The Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \text{)}$$

(2) (25 points) A particle of mass m moves in a one-dimensional potential

$$V(x) = -g\frac{\hbar^2}{m}\,\delta(x^2 - a^2)\,,$$

where g is a positive dimensionless constant, $\delta(x)$ the Dirac delta function, and a a positive constant. Determine the number of bound states for the particle within the potential according to the parameter g.

(3) Two non-interacting identical spin-1/2 particles of mass m are stored inside a one-dimensional harmonic potential

$$V(x) = \frac{1}{2} m\omega^2 x^2 ,$$

where ω is a positive constant. The single-particle energy eigenfuctions for the one-dimensional simple harmonic potential are

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(y - \frac{d}{dy}\right)^n e^{-y^2/2}\Big|_{y=\sqrt{\frac{m\omega}{\hbar}}x}$$

with the corresponding eigenenergies

$$E_n(x) = \left(n + \frac{1}{2}\right)\hbar\omega, \qquad\qquad n = 0,1,2,3 \ldots$$

(a) (10 points) Write down the energy and the total wavefunction(s) for the ground state of the two-particle system.

(b) (15 points) Write down the energy and the total wavefunction(s) for the first excited state of the two-particle system.
(*Hint*: You must take into account the spin degrees of freedom and all possible degeneracies.)

(4) (25 points) A particle of mass m is initially at the ground state of the one-dimensional infinite square-well potential

$$V(x) = \begin{cases} 0, & \text{if } 0 \le x \le a, \\ \infty, & \text{otherwise,} \end{cases}$$

where a is a positive constant. Suppose at time $t = 0$, an additional potential is switched on

$$\Delta V(x) = \begin{cases} V_0, & \text{if } 0 \le x \le a/2, \\ 0, & \text{otherwise,} \end{cases}$$

where V_0 is a small positive constant (much less than the ground-state energy of the original infinite square-well). If the perturbing potential $\Delta V(x)$ is switched off at time $t = T$, find the probability for the particle to be found in the first excited state of the original infinite square-well potential.