QM–part I

1. (20 points) For a (normalized) quantum state $|\psi\rangle$, its density operator is given by $\rho = |\psi\rangle\langle\psi|$. Suppose the time evolution for the density operator can be described using the sum

$$\rho(t) = \sum_{i=1}^{N} K_i \rho(t_0) K_i^\dagger \,,$$

where t_0 is the initial time, N is a positive integer, and K_i are a set of suitable operators. If the time evolution would conserve probability (*i.e.* the state would remain "normalized" throughout), find the condition that K_i must satisfy.

2. (15 points) A particle of mass m moving along the x-axis in the presence of the potential

$$U(x) = U_0 \sin\left(\frac{2\pi x}{a}\right) ,$$

where U_0 and a are positive constants. Determine whether the momentum of the particle would be conserved or not during the motion.

3. (15 points) A particle of mass m moving along the x-axis under an attractive potential

$$V(x) = -V_0\, \delta(x)$$

with V_0 a positive constant. Making use of momentum-space representation, find the ground-state energy and wavefunction for the particle.

QM–part II

4. (20 points) Consider a system of two spin-1/2 particles. The orbital degrees of freedom of this system are ignored. If $\vec{S}_1$ and $\vec{S}_2$ are their spin operators and $\vec{S} = \vec{S}_1 + \vec{S}_2$, the two-particle Hilbert space $V_{1\otimes 2}$ is spanned by the four vectors $|s_1 = 1/2, m_1\rangle \otimes |s_2 = 1/2, m_2\rangle$ where $m_1 = \pm 1/2$ and $m_2 = \pm 1/2$. We use $|1/2, 1/2\rangle \otimes |1/2, 1/2\rangle \rightarrow (1,0,0,0)^T$, $|1/2, 1/2\rangle \otimes |1/2, -1/2\rangle \rightarrow (0,1,0,0)^T$, $|1/2, -1/2\rangle \otimes |1/2, 1/2\rangle \rightarrow (0,0,1,0)^T$, $|1/2, -1/2\rangle \otimes |1/2, -1/2\rangle \rightarrow (0,0,0,1)^T$, where T is the transport. Prove that (a)

$$S_z \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and (b)

$$S^2 \rightarrow \hbar^2 \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

5. (20 points) Consider the perturbative solution to a class of phenomena described by $H(t) = H^0 + H^1(t)$ where H^0 is a time-independent piece whose eigenvalue problem has been solved and H^1 is a small time-dependent perturbation. The eigenkets $|n^0\rangle$ with eigenvalue E_n^0 of H^0 form a complete basis. If the eigenket of H is $|\Psi(t)\rangle = \Sigma_n d_n(t) e^{-iE_n^0 t/\hbar}|n^0\rangle$ and the system is in the state $|i^0\rangle$ at $t = 0$, prove that (a)

$$\frac{d}{dt} d_f = 0$$

for the zeroth order approximation and (b)

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_0^t \langle f^0|H^1(t')|i^0\rangle e^{i\frac{(E_f^0 - E_i^0)t'}{\hbar}} dt'$$

for the first order approximation.

6. (10 points) Show that $\wp_1 = \frac{3}{4}I + \vec{S}_1 \cdot \vec{S}_2/\hbar^2$ and $\wp_0 = \frac{1}{4}I - (\vec{S}_1 \cdot \vec{S}_2)/\hbar^2$ are projection operators, i.e., obey $\wp_i \wp_j = \delta_{ij}\wp_j$. [Hint: A very useful indentity $(\vec{A} \cdot \vec{\sigma})(\vec{B} \cdot \vec{\sigma}) = \vec{A} \cdot \vec{B} I + i(\vec{A} \times \vec{B}) \cdot \vec{\sigma}$]