1. For a one-dimensional quantum system with hamiltonian $H = P^2/2m + V(x)$,

(1a) (10%) Show that the conservation of probability can be expressed as the continuity equation

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial j(x,t)}{\partial x} = 0, \quad j = \frac{\hbar}{2mi}(\Psi^* \partial_x \Psi - \Psi \partial_x \Psi^*).$$

(1b) (10%) If $V(x) = V_R(x) - iV_I$, show that the total probability for finding the particle decreases exponentially as $\exp(-2V_I t/\hbar)$.

2. (2a) (10%) Let E_0 be the exact ground state energy of the hamiltonian H. Show that, for any trial state $|\Psi>$

$$\frac{< \Psi|H|\Psi >}{< \Psi|\Psi >} \geq E_0.$$

(2b) (15%) Using the trial wavefunction $\Psi = e^{-\alpha x^2/2}$ to estimate the ground state energy of a one-dimensional problem with $H = P^2/2m + V$ where $V(x) = -aV_0\delta(x)$.

3. (3a) (10%) Show that

$$(\vec{A} \cdot \sigma)(\vec{B} \cdot \sigma) = (\vec{A} \cdot \vec{B})I + i(\vec{A} \times \vec{B}) \cdot \sigma.$$

(3b) (10%) Show that

$$e^{-i\theta \hat{n}\cdot\sigma/2} = \cos(\theta/2)I - i\sin(\theta/2)\hat{n} \cdot \sigma.$$

(3c) (10%) Let

$$|z+ >= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\hat{n}+ >= \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix}$$

where $\hat{n}$ points in the direction (θ, ϕ). Show that $|\hat{n}+>$ can be obtained from $|z+>$ by rotations.

4. For a two-dimensional electron in a magnetic field $\mathbf{B} = B\hat{z}$, the Hamiltonian is given by

$$H_{em} = \frac{(\mathbf{P} + \frac{e}{c}\mathbf{A})^2}{2m}.$$

(4a) (10%) Show that

$$[mv_x, mv_y] = \frac{\hbar e B}{ic}$$

where $mv_i \equiv P_i + \frac{e}{c}A_i$.

(4b) (15%) Using (4a) to derive the energy spectrum of H_{em} (i.e the Landau levels).

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}I$$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = (\pi/a)^{1/2}.$$